

# RECONSTRUCTING VECTOR BUNDLES ON CURVES FROM THEIR DIRECT IMAGE ON SYMMETRIC POWERS

INDRANIL BISWAS AND D. S. NAGARAJ

**ABSTRACT.** Let  $C$  be an irreducible smooth complex projective curve, and let  $E$  be an algebraic vector bundle of rank  $r$  on  $C$ . Associated to  $E$ , there are vector bundles  $\mathcal{F}_n(E)$  of rank  $nr$  on  $S^n(C)$ , where  $S^n(C)$  is the  $n$ -th symmetric power of  $C$ . We prove the following: Let  $E_1$  and  $E_2$  be two semistable vector bundles on  $C$ , with  $\text{genus}(C) \geq 2$ . If  $\mathcal{F}_n(E_1) \simeq \mathcal{F}_n(E_2)$  for a fixed  $n$ , then  $E_1 \simeq E_2$ .

## 1. INTRODUCTION

Let  $C$  be an irreducible smooth projective curve defined over the field of complex numbers. Let  $E$  be a vector bundle of rank  $r$  on  $C$ . Let  $S^n(C)$  be  $n$ -th symmetric power of  $C$ . Let  $q_1$  (respectively,  $q_2$ ) be the projection of  $S^n(C) \times C$  to  $S^n(C)$  (respectively,  $C$ ). Let  $\Delta_n \subset S^n(C) \times C$  be the universal effective divisor of degree  $n$ . The direct image

$$\mathcal{F}_n(E) := q_{1*}(q_2^*(E)|_{\Delta_n})$$

is a vector bundle of rank  $nr$  over  $S^n(C)$ . These vector bundles  $\mathcal{F}_n(E)$  are extensively studied (see [1], [2], [3], [4], [5]).

Assume that  $\text{genus}(C) \geq 2$ . We prove the following (see Theorem 3.2):

**Theorem 1.1.** *Let  $E_1$  and  $E_2$  be semistable vector bundles on  $C$ . If the two vector bundles  $\mathcal{F}_n(E_1)$  and  $\mathcal{F}_n(E_2)$  on  $S^n(C)$  are isomorphic for a fixed  $n$ , then  $E_1$  is isomorphic to  $E_2$ .*

## 2. PRELIMINARIES

Fix an integer  $n \geq 2$ . Let  $S_n$  be the group of permutations of  $\{1, \dots, n\}$ . Given an irreducible smooth complex projective curve  $C$ , the group  $S_n$  acts on  $C^n$ , and the quotient  $S^n(C) := C^n/S_n$  is an irreducible smooth complex projective variety of dimension  $n$ .

An effective divisor of degree  $n$  on  $C$  is a formal sum of the form  $\sum_{i=1}^r n_i z_i$ , where  $z_i$  are points on  $C$  and  $n_i$  are positive integers, such that  $\sum_{i=1}^r n_i = n$ . The set of all effective divisors of degree  $n$  on  $C$  is naturally identified with  $S^n(C)$ .

Let  $q_1$  (respectively,  $q_2$ ) be the projection of  $S^n(C) \times C$  on to  $S^n(C)$  (respectively,  $C$ ). Define

$$\Delta_n := \{(D, z) \in S^n(C) \times C \mid z \in D\} \subset S^n(C) \times C.$$

Then  $\Delta_n$  is a smooth hypersurface on  $S^n(C) \times C$ ; it is called the *universal effective divisor* of degree  $n$  of  $C$ . The restriction of the projection  $q_1$  to  $\Delta_n$  is a finite morphism

$$(2.1) \quad q := q_1|_{\Delta_n} : \Delta_n \longrightarrow S^n(C)$$

---

2000 *Mathematics Subject Classification.* 14J60, 14C20.

*Key words and phrases.* Symmetric power, direct image, curve.

of degree  $n$ .

Let  $E$  be a vector bundle on  $C$  of rank  $r$ . Define

$$\mathcal{F}_n(E) := q_*(q_2^*(E)|_{\Delta_n})$$

to be the vector bundle on  $S^n(C)$  of rank  $nr$ .

The *slope*  $E$  is defined to be  $\mu(E) := \text{degree}(E)/r$ . The vector bundle  $E$  is said to be *semistable* if  $\mu(F) \leq \mu(E)$  for every nonzero subbundle  $F$  of  $E$ .

### 3. THE RECONSTRUCTION

Henceforth, we assume that  $\text{genus}(C) \geq 2$ .

We first consider the case of  $n = 2$ . The hypersurface  $\Delta_2$  in  $S^2(C) \times C$  can be identified with  $C \times C$ . In fact the map  $(x, y) \mapsto (x + y, x)$  is an isomorphism from  $C \times C$  to  $\Delta_2$  (cf. [3]). Let

$$q : \Delta_2 \longrightarrow S^2(C)$$

be the map in (2.1). Under the above identification of  $\Delta_2$  with  $C \times C$ , the map  $q$  coincides with the quotient map

$$(3.1) \quad f : C \times C \longrightarrow S^2(C) = (C \times C)/S_2.$$

For  $i = 1, 2$ , let

$$p_i : C \times C \longrightarrow C$$

be the projection to the  $i$ -th factor. The diagonal  $\Delta \subset C \times C$  is canonically isomorphic to  $C$ , and hence any vector bundle on  $C$  can also be thought of as a vector bundle on  $\Delta$ . For a vector bundle  $E$  on  $C$ , we have the short exact sequence

$$0 \longrightarrow V(E) := f^*\mathcal{F}_2(E) \longrightarrow p_1^*E \oplus p_2^*E \xrightarrow{q} E \longrightarrow 0,$$

where  $f$  is defined in (3.1), and  $q$  is the homomorphism defined by  $(u, v) \mapsto u - v$  (cf. [3]). Let

$$\phi_i : V(E) \longrightarrow p_i(E), \quad i = 1, 2,$$

be the restriction of the projection  $p_1^*E \oplus p_2^*E \longrightarrow p_i^*E$  to  $V(E) \subset p_1^*E \oplus p_2^*E$ . We have the following two exact sequences:

$$(3.2) \quad 0 \longrightarrow (p_2^*E)(-\Delta) := (p_2^*E) \otimes \mathcal{O}_{C \times C}(-\Delta) \longrightarrow V(E) \xrightarrow{\phi_1} p_1^*E \longrightarrow 0$$

(we are using the fact that the restriction of the line bundle  $\mathcal{O}_{C \times C}(-\Delta)$  to  $\Delta$  is  $K_\Delta = K_C$ , where  $K_\Delta$  and  $K_C$  are the canonical line bundles of  $\Delta$  and  $C$  respectively) and

$$0 \longrightarrow (p_1^*E)(-\Delta) := (p_1^*E) \otimes \mathcal{O}_{C \times C}(-\Delta) \longrightarrow V(E) \xrightarrow{\phi_2} p_2^*E \longrightarrow 0.$$

**Proposition 3.1.** *Let  $E$  and  $F$  be two semistable vector bundles on  $C$  such that  $\mathcal{F}_2(E) \simeq \mathcal{F}_2(F)$ . Then  $E$  is isomorphic to  $F$ .*

*Proof.* The restriction of the exact sequence in (3.2) to the diagonal  $\Delta = C$  gives a short exact sequence of vector bundles on  $C$ :

$$(3.3) \quad 0 \longrightarrow E \otimes K_C \longrightarrow J^1(E) \longrightarrow E \longrightarrow 0,$$

where  $K_C$  is the canonical bundle on  $C$ . Similarly we have a short exact sequence

$$(3.4) \quad 0 \longrightarrow F \otimes K_C \longrightarrow J^1(F) \longrightarrow F \longrightarrow 0.$$

Since  $\mathcal{F}_2(E) \simeq \mathcal{F}_2(F)$ , we see that  $J^1(E) \simeq J^1(F)$ . As  $E$  (respectively,  $F$ ) is semistable, and  $\text{degree}(K_C) > 0$ , the subbundle  $E \otimes K_C$  (respectively,  $F \otimes K_C$ ) of  $J^1(E)$  (respectively,  $J^1(F)$ ) in (3.3) (respectively, (3.4)) is the first term in the Harder–Narasimhan filtration of  $J^1(E)$  (respectively,  $J^1(F)$ ). Since  $J^1(E) \simeq J^1(F)$ , this implies that  $E \simeq F$ .  $\square$

Now we consider the general case of  $n \geq 3$ .

**Theorem 3.2.** *Let  $E$  and  $F$  be semistable vector bundles on  $C$  such that  $\mathcal{F}_n(E) \simeq \mathcal{F}_n(F)$ . Then the vector bundle  $E$  is isomorphic to  $F$ .*

*Proof.* The universal effective divisor  $\Delta_n \subset S^n(C) \times C$  of degree  $n$  can be identified with  $S^{n-1}(C) \times C$  using the morphism

$$f : S^{n-1}(C) \times C \longrightarrow \Delta_n$$

that sends any  $(D, z) \in S^{n-1}(C) \times C$  to  $(D + z, z)$ . The composition

$$(3.5) \quad S^{n-1}(C) \times C \xrightarrow{f} \Delta_n \xrightarrow{q} S^n(C),$$

where  $q$  is defined in (2.1), will be denoted by  $\bar{q}$ . We note that

$$\mathcal{F}_n(E) = \bar{q}_* p_2^* E,$$

where  $p_2 : S^{n-1}(C) \times C \longrightarrow C$  is the natural projection.

Let  $f_1 : S^{n-1}(C) \times C \longrightarrow S^{n-1}(C)$  is the natural projection. Let

$$\alpha : C \times C \longrightarrow S^{n-1}(C) \times C$$

be the morphism defined by  $(x, y) \mapsto ((n-1)x, y)$ . Then the pullback  $(\bar{q} \circ \alpha)^*(\mathcal{F}_n(E))$ , where  $\bar{q}$  is constructed in (3.5), fits in an exact sequence:

$$(3.6) \quad 0 \longrightarrow p_2^* E \otimes \mathcal{O}_{C \times C}(-(n-1)\Delta) \longrightarrow (\bar{q} \circ \alpha)^*(\mathcal{F}_n(E)) \longrightarrow (f_1 \circ \alpha)^* \mathcal{F}_{n-1}(E) \longrightarrow 0,$$

where  $f_1$  is defined above. The above projection  $(\bar{q} \circ \alpha)^*(\mathcal{F}_n(E)) \longrightarrow (f_1 \circ \alpha)^* \mathcal{F}_{n-1}(E)$  follows from the fact that  $f_1 \circ \alpha(z) \subset \bar{q} \circ \alpha(z)$  for any  $z \in C \times C$ .

Define the vector bundle

$$J^{n-1}(E) := (f \circ \alpha)^*(\mathcal{F}_n(E))|_{\Delta} \longrightarrow \Delta = C$$

on the diagonal in  $C \times C$ . Restricting the exact sequence in (3.6) to  $\Delta$ , we get a short exact sequence of vector bundles

$$0 \longrightarrow J^{n-2}(E) \otimes K_C \longrightarrow J^{n-1}(E) \longrightarrow E \longrightarrow 0;$$

note that  $J^0(E) = E$ .

Therefore, by induction on  $n$ , we get a filtration of subbundles of  $J^{n-1}(E)$

$$(3.7) \quad 0 = W_n \subset W_{n-1} \subset \cdots \subset W_1 \subset W_0 = J^{n-1}(E),$$

such that  $W_j/W_{j+1} = E \otimes K_C^{\otimes j}$  for all  $j \in [0, n-1]$ . In particular,  $W_{n-1} = E \otimes K_C^{\otimes(n-1)}$ .

Since  $E$  is semistable, and  $\text{degree}(K_C) > 0$ , we conclude that  $E \otimes K_C^{\otimes j}$  is semistable for all  $j \in [0, n-1]$ , and

$$\mu(W_j/W_{j+1}) < \mu(W_{j+1}/W_{j+2})$$

for all  $j \in [0, n-2]$ . Consequently, the filtration of  $J^{n-1}(E)$  in (3.7) coincides with the Harder–Narasimhan filtration of  $J^{n-1}(E)$ . In particular, the first term of the Harder–Narasimhan filtration (the maximal semistable subsheaf) of  $J^{n-1}(E)$  is the subbundle  $E \otimes K_C^{\otimes(n-1)}$ .

Using this, and the fact that  $J^{n-1}(E) \simeq J^{n-1}(F)$  (recall that  $\mathcal{F}_n(E) \simeq \mathcal{F}_n(F)$ ), we conclude that  $F$  is isomorphic to  $E = W_0/W_1$ .  $\square$

## REFERENCES

- [1] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris, *Geometry of algebraic curves*, Vol. I, Grundlehren der Mathematischen Wissenschaften, vol. 267, Springer-Verlag, New York, 1985.
- [2] I. Biswas and F. Laytimi, Direct image and parabolic structure on symmetric product of curves, *Jour. Geom. Phys.* **61** (2011), 773–780.
- [3] I. Biswas and A.J. Parameswaran, Vector bundles on symmetric product of a curve, *Jour. Ramanujan Math. Soc.* **26** (2011), 351–355.
- [4] A. El Mazouni, F. Laytimi and D. S. Nagaraj, Secant bundles on second symmetric power of a curve, *Jour. Ramanujan Math. Soc.* **26** (2011), 181–194.
- [5] R.L.E. Schwarzenberger, Vector bundles on the projective plane, *Proc. London Math. Soc.* **11** (1961), 623–640.

SCHOOL OF MATHEMATICS, TATA INSTITUTE OF FUNDAMENTAL RESEARCH, HOMI BHABHA ROAD, MUMBAI 400005, INDIA

*E-mail address:* `indranil@math.tifr.res.in`

THE INSTITUTE OF MATHEMATICAL SCIENCES, CIT CAMPUS, TARAMANI, CHENNAI 600113, INDIA

*E-mail address:* `dsn@imsc.res.in`